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Rathkjen, Arne

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A. Rathkjen
Department of Building Technology and
Structural Engineering
University of Aalborg
DK-9000 Aalborg, Denmark

Abstract

A state of plane stress is illustrated by means of two families of curves, each family representing constant values of a derivative of Airy's stress function. The two families of curves form a map giving in the first place an overall picture of regions of high and low stress, and in the second place, the map comprises a complete graphic representation of the stress at any point.

1. INTRODUCTION

Several systems of curves are used to describe the stress state in structural elements submitted to plane stress. Among those most common in use principal stress trajectories, isochromatics and lines of equal von Mises stress have to be mentioned. The stress trajectories give the directions of the principal stress components at any point while information concerning the magnitude of the stress components can be deduced in very special situations only. Isochromatics, the fundamental lines in photoelasticity, give the value of the maximum shear stress component at any point, and the lines of equal von Mises stress, often used in connection with numerical computations and also quite often in combination with a colour code, give the values of a rather special combination of stress components. Each of the three systems comprises valuable information, but taken separately none of them are capable of giving a complete picture of the stress state.

The force lines to be described in the following sections give three stress components and thus a complete representation of the stress state at any point. The three components refer to a Cartesian coordinate system, rectangular or skew, and in this respect it has to be admitted that the three systems mentioned above have an advantage in that they are independent of any coordinate system.

The basic equations are derived in section 2, showing that the stress components are closely related to the components of two gradient vectors, some examples illustrating the stress around elliptical holes are shown in section 3, and in section 4 and the appendix some rules for reading off information from the map made up of force lines are given.

2. BASIC EQUATIONS

Given a scalar quantity φ that is a function of position in space, the gradient of φ is a vector that is also a function of position. In two dimensions φ is a function of position in a plane and a map of curves φ = constants with equal distance $\Delta\varphi$ may be drawn to show the variation of φ over the plane. The gradient vector in this case, is a plane vector which at any point has a direction orthogonal to the particular curve φ = constant through that point, and a magnitude inversely proportional to the distance between the neighbouring curves. In a rectangular cartesian x, y, z-coordinate system with unit base vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$, the scalar $\varphi = \varphi(x, y, z)$ is a function of x, y and z, and the gradient is $\mathbf{grad}\varphi = \hat{\mathbf{i}}\partial\varphi/\partial x + \hat{\mathbf{j}}\partial\varphi/\partial y + \hat{\mathbf{k}}\partial\varphi/\partial z$. In two dimensions this expression reduces to $\mathbf{grad}\varphi = \hat{\mathbf{i}}\partial\varphi/\partial x + \hat{\mathbf{j}}\partial\varphi/\partial y$.

The stress components in plane stress, σ_{xx}, σ_{xy} and σ_{yy} are components of *stress vectors*

$$\begin{aligned} \bar{\boldsymbol{\sigma}}_x &= \sigma_{xx}\hat{\mathbf{i}} + \sigma_{xy}\hat{\mathbf{j}} \\ \bar{\boldsymbol{\sigma}}_y &= \sigma_{xy}\hat{\mathbf{i}} + \sigma_{yy}\hat{\mathbf{j}} \end{aligned} \tag{2.1}$$

acting on planes with unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$, respectively, as normals. The task now is to relate these stress vectors to some gradient vectors.

To establish such relations the stress components are expressed in terms of Airy's stress function $U(x, y)$ as

$$\sigma_{xx} = \partial^2 U/\partial y^2 \quad , \quad \sigma_{xy} = -\partial^2 U/\partial x\partial y \quad , \quad \sigma_{yy} = \partial^2 U/\partial x^2 \tag{2.2}$$

thus satisfying the equations of equilibrium when body forces are absent. Introducing the notation

$$\xi(x, y) = \partial U/\partial x \quad , \quad \eta(x, y) = \partial U/\partial y \tag{2.3}$$

we have

$$\sigma_{xx} = \partial\eta/\partial y \quad , \quad \sigma_{xy} = -\partial\xi/\partial y = -\partial\eta/\partial x \quad , \quad \sigma_{yy} = \partial\xi/\partial x \tag{2.4}$$

so that the stress vectors are

$$\begin{aligned} \bar{\boldsymbol{\sigma}}_x &= \hat{\mathbf{i}}\partial\eta/\partial y - \hat{\mathbf{j}}\partial\eta/\partial x = \mathbf{grad}\eta \times \hat{\mathbf{k}} \\ \bar{\boldsymbol{\sigma}}_y &= -\hat{\mathbf{i}}\partial\xi/\partial y + \hat{\mathbf{j}}\partial\xi/\partial x = \hat{\mathbf{k}} \times \mathbf{grad}\xi \end{aligned} \tag{2.5}$$

where $\hat{\mathbf{k}}$ is the normal to the x, y-plane as above.

This means that the stress vectors are not orthogonal to the curves ξ = constants and η = constants, as gradient vectors usually are, rather they are parallel with them and the curves give the directions of the stress vectors directly. The magnitude of each stress vector, however, is that of the gradient vector, inversely proportional to the distance between the curves.

In figure 2.1 ξ,η-lines are shown illustrating the stress distribution in a beam, supported at the end surfaces and loaded with a uniform stress at the upper surface. An $\hat{\mathbf{i}}$-vector is indicated with a thin line and a $\hat{\mathbf{j}}$-vector with a heavy line. Correspondingly, the curves representing stress vectors on planes with $\hat{\mathbf{i}}$ as normal, are drawn with thin lines, and the curves representing stress vectors on planes with $\hat{\mathbf{j}}$ as normal, are drawn with heavy lines. Further, a full line stands for tension and a broken line for compression. Thin lines are used for the family with the highest number of curves.

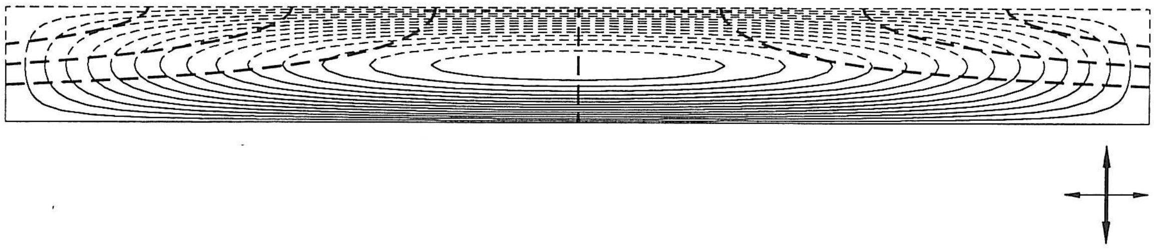


Figure 2.1 Stress in a beam.

The connection between this kind of stress representation and the usual illustration of stress distribution along a section $A-A$ is shown in figure 2.2.

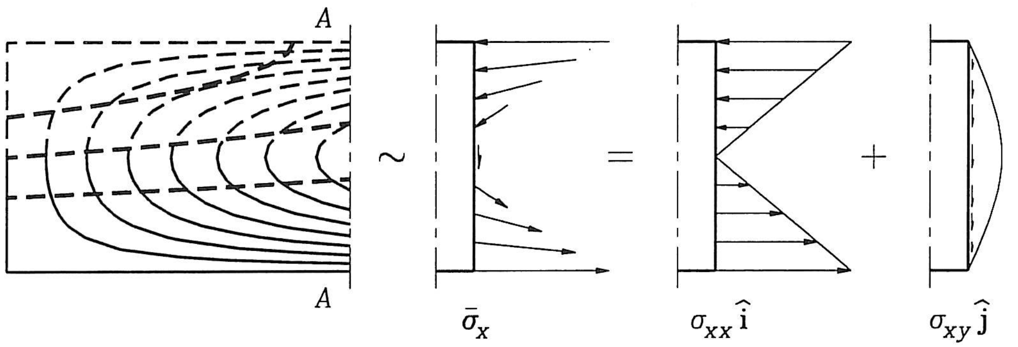


Figure 2.2 Stress along $A-A$.

Lines of constant values of ξ and η are *force lines*, representing resultant force vectors, see Muskhelishvili (1963, p.115) and Timoshenko & Goodier (1951, p.190). They are used by Michell (1900) to illustrate some of his solutions, but unfortunately Michell gives a very complicated description of what is here shown to be in fact just two simple gradient fields. Also Michell presents two maps for each solution, thus missing the information that, as shown in section 4, may be deduced from the knowledge of the angle ψ, under which ξ- and η-lines intersect.

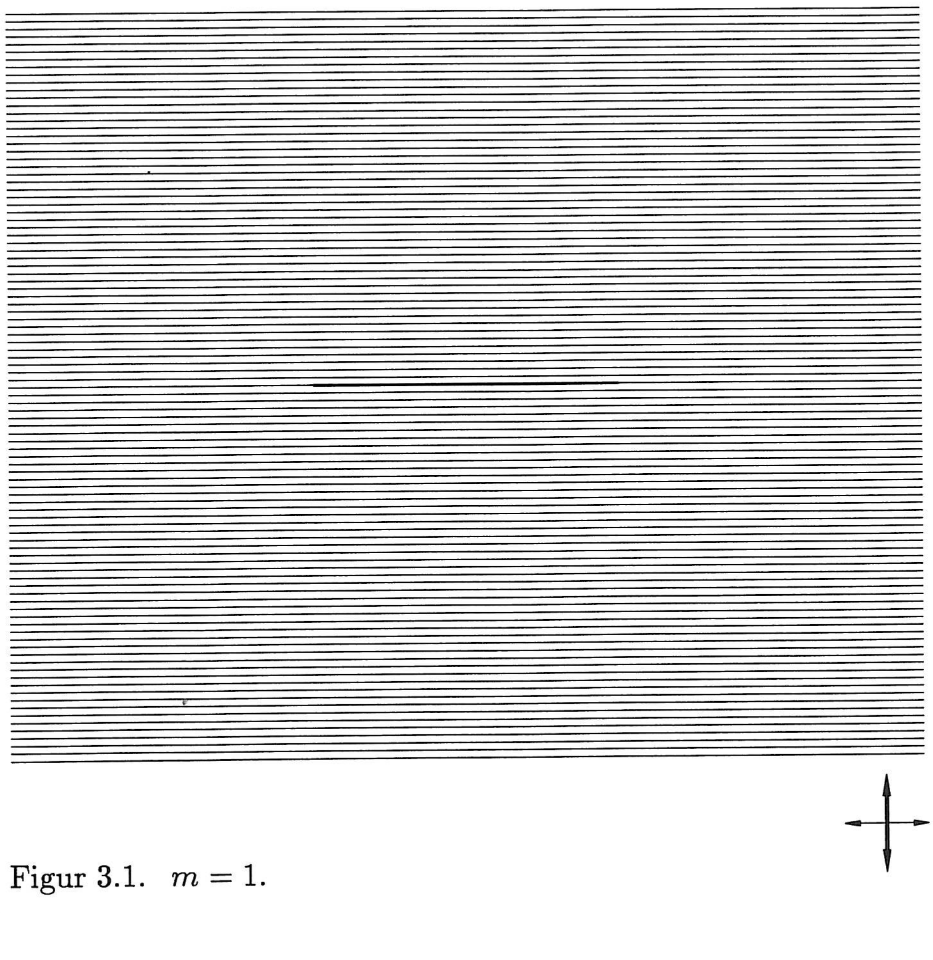
3. EXAMPLES

Figures 3.1 to 3.5 show the stress distributions around elliptical holes in infinite plates, isotropic and linear elastic, loaded in uniaxial tension. The tension is in the $\hat{\mathbf{i}}$-direction as is also one of the semiaxes of the ellipses. The lengths of the semiaxes are a and b and $m = (a - b)/(a + b)$ is a parameter such that

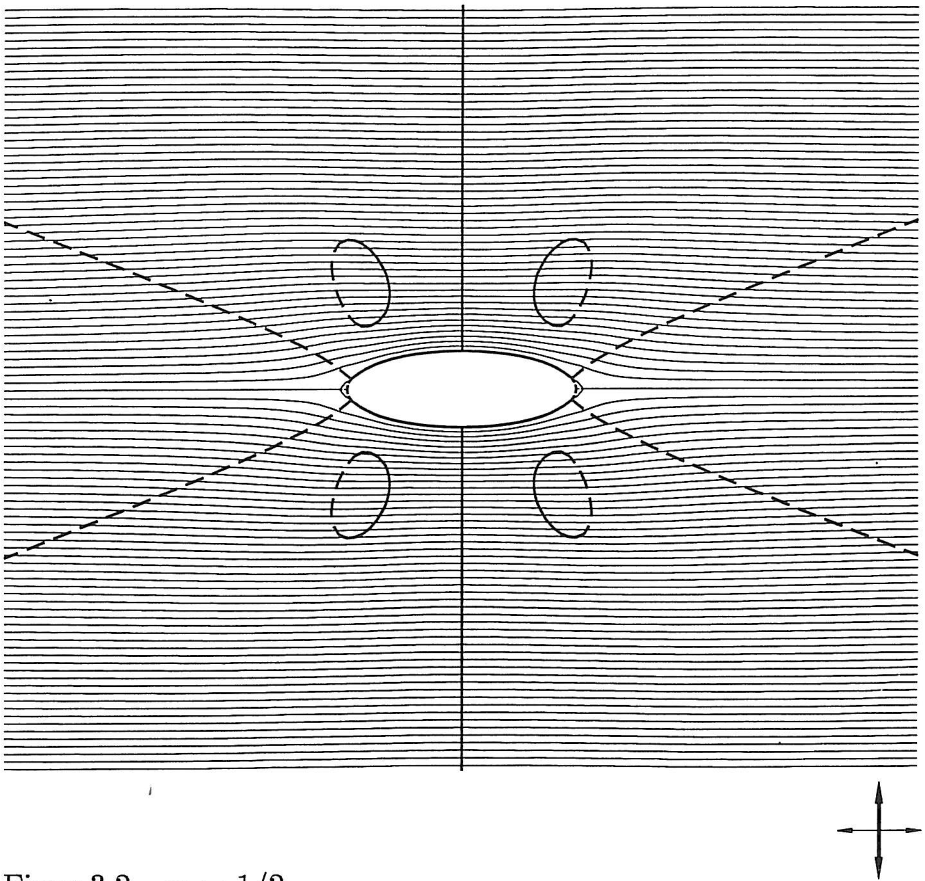
$$\begin{aligned} a &= R(1+m) \\ b &= R(1-m) \end{aligned} \tag{3.1}$$

where R is a scaling factor. Each example is part of a sequence in which m has the values 1, 1/2, 0, −1/2 and −1. With $m = 1$ the ellipse is a slit in the same direction as the tension, with $m = 1/2$ the ellipse has its major semiaxis in that direction, $m = 0$ reduces the ellipse to a circle while $m = -1/2$ and $m = -1$ correspond to an ellipse and a slit with major semiaxes orthogonal to the direction of the tension.

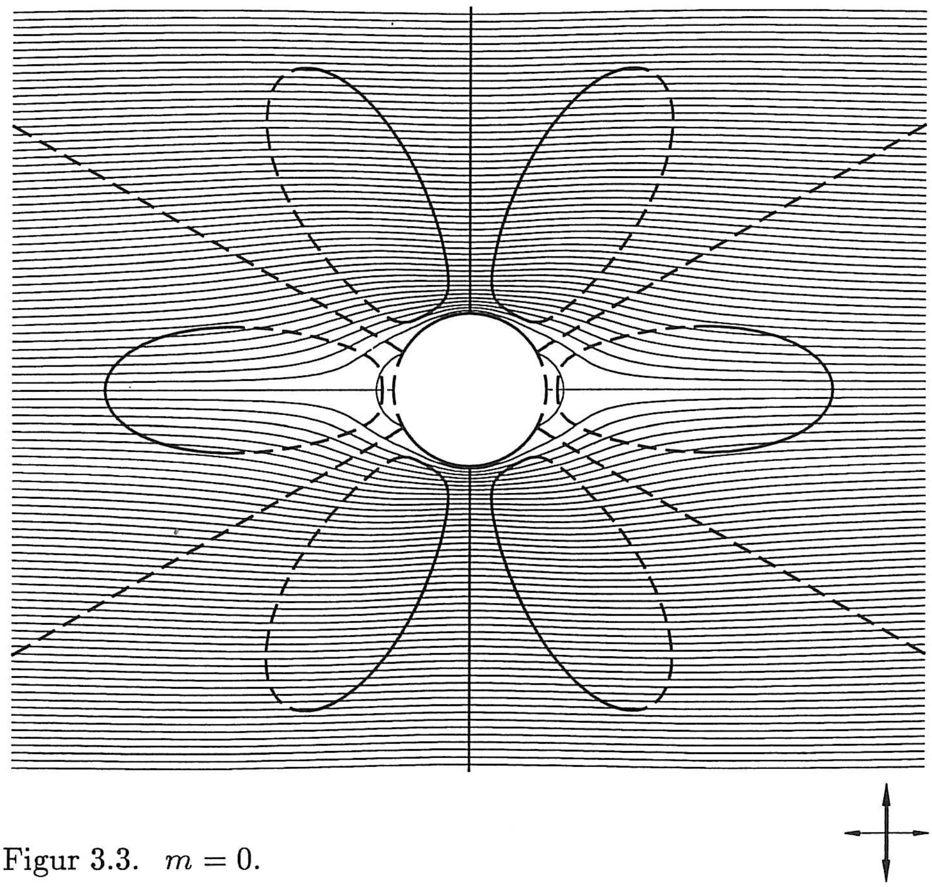
The figures illustrate how the stress field changes from no disturbance at all to infinitely high stress concentrations.



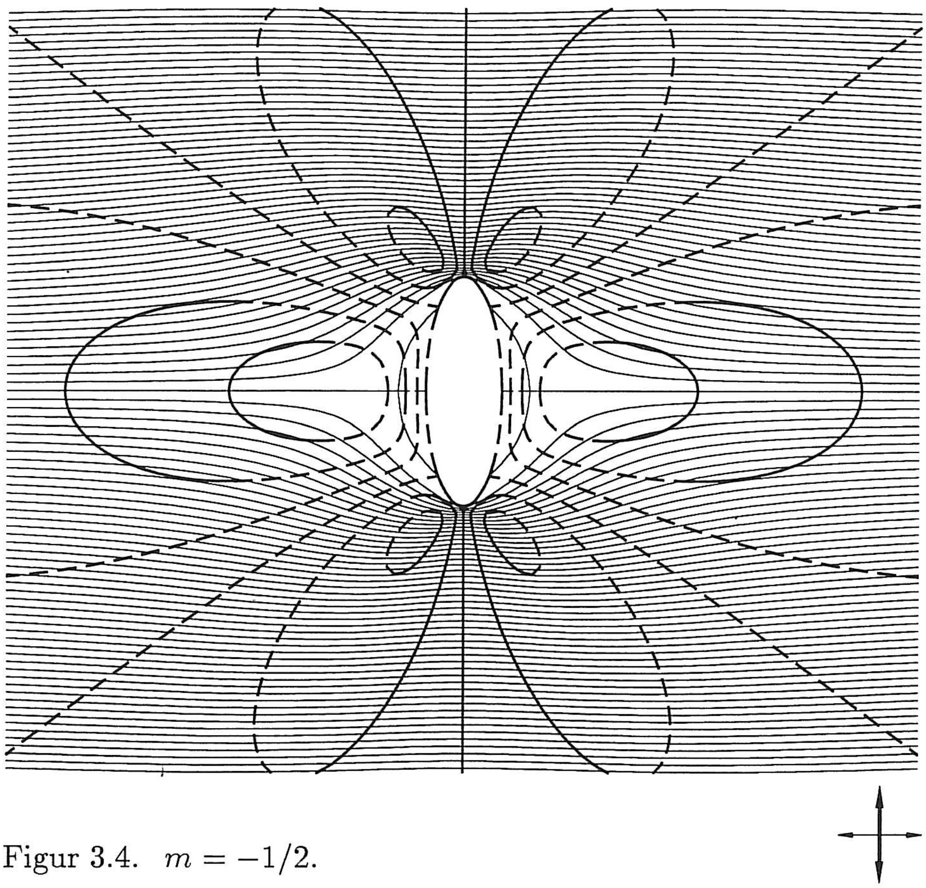
Figur 3.1. $m = 1$.



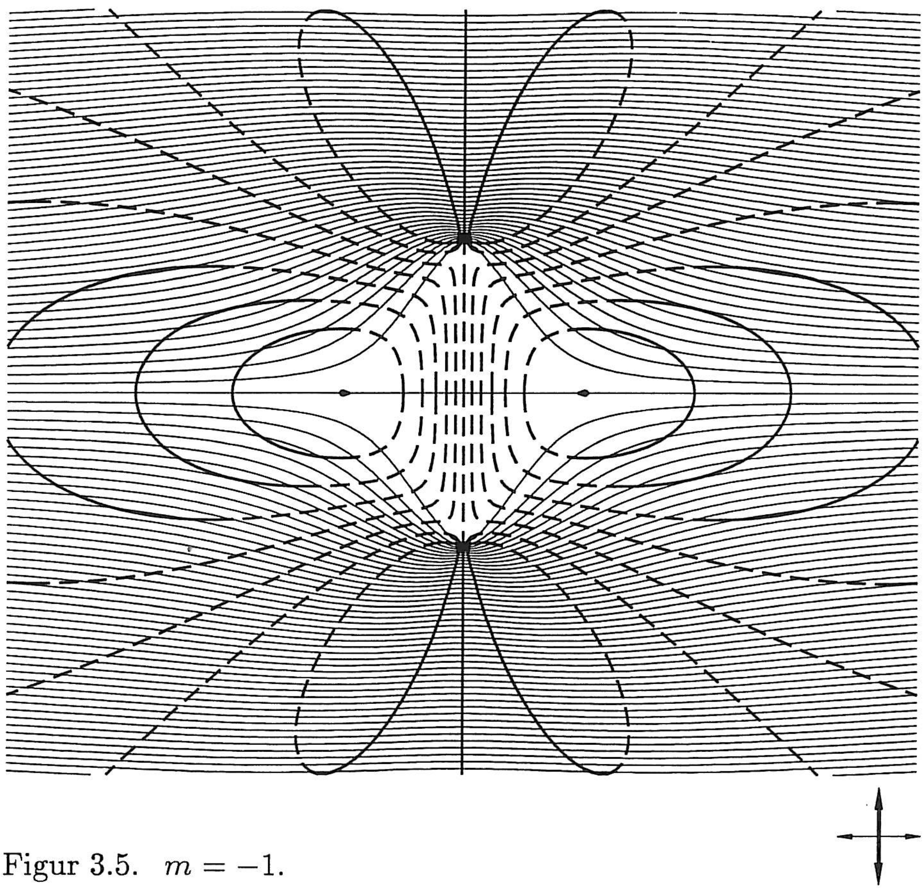
Figur 3.2. $m = 1/2$.



Figur 3.3. $m = 0$.



Figur 3.4. $m = -1/2$.



Figur 3.5. $m = -1$.

The solutions illustrated here are elasticity solutions based on the representation

$$\xi + i\eta = \partial U/\partial x + i\partial U/\partial y = \varphi + z\overline{\varphi}' + \overline{\psi} \tag{3.2}$$

where $\varphi(z)$ and $\psi(z)$ are complex potentials in Muskhelishvili's (1963) notation.

4. RELATIONS BETWEEN GEOMETRIC AND DYNAMIC QUANTITIES

A few lines ξ = constants and η = constants are shown in figure 4.1. The lines may be regarded as coordinate lines in a curvilinear coordinate system, and the coordinate

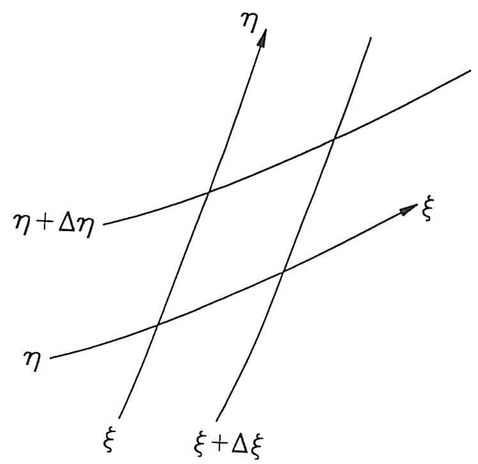


Figure 4.1. ξ, η-system.

distance between neighbouring lines is $\Delta\xi = \Delta\eta$ = constant. The geometric distances between the lines are shown in figure 4.2 where the parallelogram in figure 4.1 has been magnified. Measured along the ξ- and η-lines these distances are denoted α and β, respectively, and measured along the normals to these curves they are a and b with

$$\begin{aligned} a &= \alpha \sin\psi \\ b &= \beta \sin\psi \end{aligned} \tag{4.1}$$

ψ being the angle between the ξ- and η-lines. The area A of the parallelogram is

$$A = \alpha\beta \sin\psi = ab/\sin\psi \tag{4.2}$$

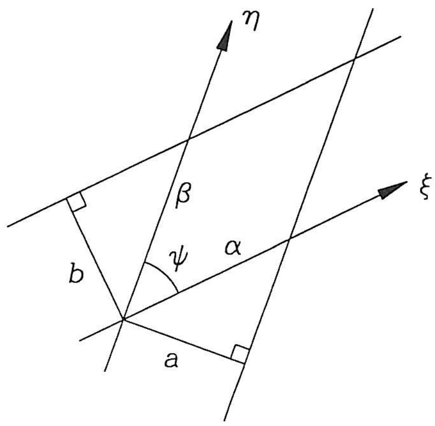


Figure 4.2. Elementary parallelogram.

The stress vector $\overline{\boldsymbol{\sigma}}_x$ is expressed as $\sigma_x \hat{\boldsymbol{\sigma}}_x$ where σ_x is the magnitude of the vector and $\hat{\boldsymbol{\sigma}}_x$ is a unit vector in the direction of that stress vector, i.e. the ξ-direction. Similar remarks are valid for $\overline{\boldsymbol{\sigma}}_y = \sigma_y \hat{\boldsymbol{\sigma}}_y$. The magnitude of a gradient vector is inversely proportional to the normal distance between two curves, but since the factor of proportionality is immaterial here, we write

$$\begin{aligned} \sigma_x &= \sqrt{\sigma_{xx}^2 + \sigma_{xy}^2} = 1/b \\ \sigma_y &= \sqrt{\sigma_{xy}^2 + \sigma_{yy}^2} = 1/a \end{aligned} \tag{4.3}$$

This means that highly stressed regions are immediately identified as the regions where lines of one or both of the two families of curves are very close. While this is probably the most important information in the map, more detailed information is available as well.

In the appendix several relations between the geometrical quantities a, b and ψ and the stress components σ_{xx}, σ_{yy} and σ_{xy} or the principal stress components σ_1 and σ_2 are derived.

Figure A2 in the appendix shows how each of the stress components σ_{xx}, σ_{yy} and σ_{xy} at any point is represented on the map. Quite often however, it is easier to obtain these quantities from the computations underlying the construction of the map, and consequently other relations will be discussed.

The expression relating the angle ψ between the ξ- and η-lines and the principal stresses σ_1 and σ_2 and the shear stress σ_{xy} is

$$\tan\psi = \sigma_1\sigma_2 / \Big((\sigma_1 + \sigma_2)\sigma_{xy} \Big) \tag{4.4}$$

see (A6) and (A10). Two values, $\psi = 0$ and $\psi = \pi/2$, are of particular interest.

The angle ψ equals zero and the ξ- and η-lines are parallel, when one of the principal stresses equals zero and the state of stress is one of uniaxial tension or uniaxial compression. While this is characteristic of unloaded boundaries, it may, of course, be observed anywhere in the plane.

The value of ψ equals $\pi/2$ and the ξ- and η-lines intersect orthogonally in three instances. In the first case, the state of stress is pure shear, $\sigma_1 = -\sigma_2$, and one of the force lines indicate tension and the other compression with $a = b$. In the second case, the state of stress is isotropic, $\sigma_1 = \sigma_2$, and both force lines which are parallel with the x- and y-lines, indicate either tension or compression again with $a = b$. The angle of intersection is a right angle since $\sigma_{xy} = 0$ no matter how the x, y-system has been chosen. Finally, the force lines intersect orthogonally where the x- and y-axes incidentally concide with the principal axes of stress. In this case the force lines have the directions of the x- and y-lines, the $\hat{\mathbf{i}}$- and $\hat{\mathbf{j}}$-vectors, while nothing can be said in advance about the lengths of a and b or about tension and compression, since the principal stresses may have any values.

Just as certain values of the intersection angle ψ correspond to special states of stress, some values of the ratio $a/b = \alpha/\beta$ correspond to values and directions of the principal stresses which are easily determined. From (A12) and (A14) in the appendix σ_1, σ_2 and θ, where θ is the angle from the ξ-line to a principal direction, as shown in figure 4.3, are determined from the relations

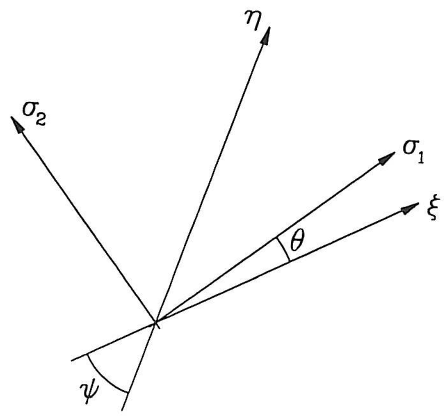


Figure 4.3. Principal directions.

$$
\begin{aligned}
&\sigma_1^2 + \sigma_2^2 = 1/a^2 + 1/b^2 \\
&\sigma_1\sigma_2 = \sin\psi/ab \\
&\tan 2\theta = b^2 \sin 2\psi/(a^2 + b^2 \cos 2\psi)
\end{aligned}
\tag{4.5}
$$

For $a \gg b$, $a = b$ and $a \ll b$ numerical values of the principal stresses and the value of θ are given in table 4.1.

	$a \gg b$	$a = b$	$a \ll b$
$\lvert\sigma_1\rvert$	$1/b$	$\sqrt{1+\cos\psi}/a$	0
$\lvert\sigma_2\rvert$	0	$\sqrt{1-\cos\psi}/a$	$1/a$
θ	0	$\psi/2$	ψ

Table 4.1. Values of $|\sigma_1|$, $|\sigma_2|$ and θ.

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Michell, J.H. (1900): *Elementary Distributions of Plane Stress*, Proc. Lond. Math. Soc., 32, 35–61.

Muskhelishvili, N.I. (1963): *Some Basic Problems of the Mathematical Theory of Elasticity*, Groningen.

Timoshenko, S. & Goodier, J.N. (1951): *Theory of Elasticity*, 2nd ed., New York.

APPENDIX

Relations between the geometric quantities a, b and ψ defined in section 4, and the stress components σ_{xx}, σ_{yy} and σ_{xy} or the principal stress components σ_1 and σ_2 are derived in this appendix.

From (2.5) the stress vectors $\overline{\boldsymbol{\sigma}}_x$ and $\overline{\boldsymbol{\sigma}}_y$ are

$$\begin{aligned} \overline{\boldsymbol{\sigma}}_x &= \mathbf{grad}\eta \times \hat{\mathbf{k}} \\ \overline{\boldsymbol{\sigma}}_y &= \hat{\mathbf{k}} \times \mathbf{grad}\xi \end{aligned} \tag{A1}$$

and from (4.3) their magnitudes are

$$\begin{aligned} \sigma_x &= 1/b \\ \sigma_y &= 1/a \end{aligned} \tag{A2}$$

In figure A1 the relations (2.1) between the stress vectors $\overline{\boldsymbol{\sigma}}_x$ and $\overline{\boldsymbol{\sigma}}_y$ and their components σ_{xx}, σ_{yy} and σ_{xy}

$$\begin{aligned} \overline{\boldsymbol{\sigma}}_x &= \sigma_{xx}\hat{\mathbf{i}} + \sigma_{xy}\hat{\mathbf{j}} \\ \overline{\boldsymbol{\sigma}}_y &= \sigma_{xy}\hat{\mathbf{i}} + \sigma_{yy}\hat{\mathbf{j}} \end{aligned} \tag{A3}$$

are shown.

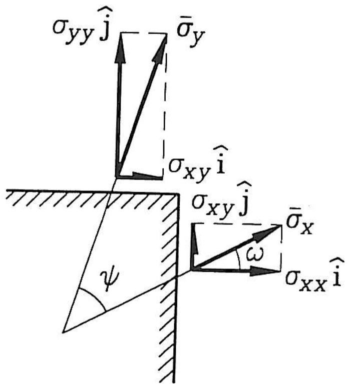

Figure A1. Stress vectors and components.

(A2) and (A3) give

$$
\begin{aligned}
\sigma_{xx} &= \cos\omega/b \\
\sigma_{xy} &= \sin\omega/b \\
\sigma_{xy} &= \cos(\omega+\psi)/a \\
\sigma_{yy} &= \sin(\omega+\psi/a
\end{aligned}
\tag{A4}
$$

where ω is the angle between the vectors $\hat{\mathbf{i}}$ and $\bar{\boldsymbol{\sigma}}_x$ (the x- and ξ-directions).

Geometric representations of these stress components in the ξ, η-system are shown in figure A2

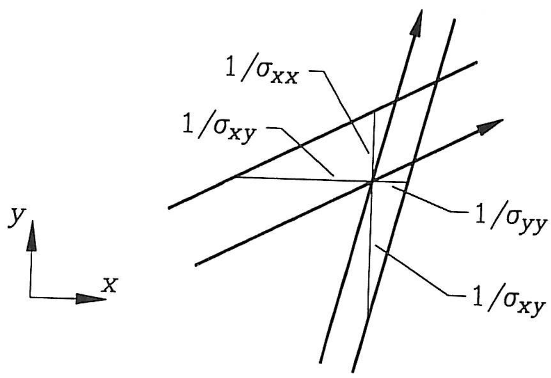

Figure A2. Graphic representation of stress components.

The relations (A4) readily give the normal distances a and b

$$
\begin{aligned}
a^2 &= 1/(\sigma_{yy}^2 + \sigma_{xy}^2) \\
b^2 &= 1/(\sigma_{xx}^2 + \sigma_{xy}^2)
\end{aligned}
\tag{A5}
$$

the angle ψ

$$\tan\psi = (\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2)/\big((\sigma_{xx} + \sigma_{yy})\sigma_{xy}\big) \tag{A6}$$

and the area $A = ab/\sin\psi$

$$A = 1/(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2) \tag{A7}$$

while the two expressions for the shear stress components give the angle ω

$$\cot\omega = \tan\psi + a/(b\cos\psi) \tag{A8}$$

Introduction of σ_1 and σ_2, the principal stress components, and the principal invariants of the stress tensor

$$\begin{aligned} I &= \sigma_1 + \sigma_2 = \sigma_{xx} + \sigma_{yy} \\ II &= \sigma_1\sigma_2 = \sigma_{xx}\sigma_{yy} - \sigma_{xy}^2 \end{aligned} \tag{A9}$$

shows that (A6) and (A7) become

$$\begin{aligned} &\tan\psi = \sigma_1\sigma_2/\big((\sigma_1 + \sigma_2)\sigma_{xy}\big) = II/(I\sigma_{xy}) \\ &A = 1/\sigma_1\sigma_2 = 1/II \end{aligned} \tag{A10}$$

Since

$$\begin{aligned} \sigma_1^2 + \sigma_2^2 &= (\sigma_1 + \sigma_2)^2 - 2\sigma_1\sigma_2 \\ &= (\sigma_{xx} + \sigma_{yy})^2 - 2(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2) \\ &= \sigma_{xx}^2 + \sigma_{yy}^2 + 2\sigma_{xy}^2 \\ &= \sigma_{yy}^2 + \sigma_{xy}^2 + \sigma_{xx}^2 + \sigma_{xy}^2 \end{aligned} \tag{A11}$$

the principal stress components are determined from

$$\begin{aligned} &\sigma_1^2 + \sigma_2^2 = 1/a^2 + 1/b^2 \\ &\sigma_1\sigma_2 = \sin\psi/ab \end{aligned} \tag{A12}$$

In a rectangular ξ, t-coordinate system as shown in figure A3, the stress components

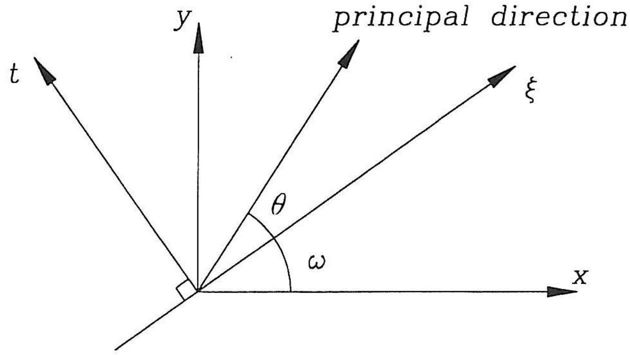

Figure A3. Principal direction.

$\sigma_{\xi\xi}$, $\sigma_{\xi t}$ and σ_{tt} are

$$\begin{aligned}
\sigma_{\xi\xi} &= \sigma_{xx}\cos^2\omega + \sigma_{yy}\sin^2\omega + 2\sigma_{xy}\sin\omega\cos\omega \\
&= \cos\omega/b + \sin\omega\cos\psi/a \\
\sigma_{\xi t} &= (\sigma_{yy} - \sigma_{xx})\sin\omega\cos\omega + \sigma_{xy}(\cos^2\omega - \sin^2\omega) \\
&= \sin\omega\sin\psi/a \\
\sigma_{tt} &= \sigma_{xx}\sin^2\omega + \sigma_{yy}\cos^2\omega - 2\sigma_{xy}\sin\omega\cos\omega \\
&= \cos\omega\sin\psi/a
\end{aligned} \tag{A13}$$

and the angle θ from the ξ-direction to the principal stress direction is given by

$$\begin{aligned}
\tan 2\theta &= 2\sigma_{\xi t}/(\sigma_{\xi\xi} - \sigma_{tt}) \\
&= b^2\sin 2\psi/(a^2 + b^2\cos 2\psi)
\end{aligned} \tag{A14}$$